

ON COMMUTATIVE ALGEBRAS OF DEGREE TWO⁽¹⁾

BY

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Let $\mathfrak{A}$ be a simple, commutative, power-associative algebra of degree 2 over an algebraically closed field $\mathfrak{F}$ of characteristic not equal to 2, 3 or 5. The degree of $\mathfrak{A}$ is defined to be the number of elements in the maximal set of pairwise orthogonal idempotents in $\mathfrak{A}$. This algebra has a unit element 1 [1, Theorem 3]. The algebras $\mathfrak{A}$ of characteristic zero were considered by Kokoris [8] and found to be Jordan algebras. Kokoris also gave examples of algebras $\mathfrak{A}$ that were not Jordan [6]. This left the problem of determining those algebras $\mathfrak{A}$ that are not Jordan algebras.

Since $1 = e + f$ where e and f are primitive orthogonal idempotents, we have a decomposition $\mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0)$ where $x \in \mathfrak{A}_e(\lambda)$ if and only if $ex = \lambda x$. We have $\mathfrak{A}_e(\lambda) = \mathfrak{A}_f(1 - \lambda)$; $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1 - \lambda) + \mathfrak{A}_e(1/2)$ for $\lambda = 1, 0$; and $\mathfrak{A}_e(1) = e\mathfrak{F} + \mathfrak{N}_1$, $\mathfrak{A}_e(0) = f\mathfrak{F} + \mathfrak{N}_0$ where $\mathfrak{N}_1$ and $\mathfrak{N}_0$ are nilideals of $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ respectively. If $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1/2)$ for $\lambda = 1, 0$ we say that e is a stable idempotent. If $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1/2) + \mathfrak{N}_{1-\lambda}$ for $\lambda = 1, 0$ we say that e is a nilstable idempotent.

The results of Albert extend the characteristic zero case to include algebras of characteristic $p \neq 2, 3, 5$ for which every idempotent is stable [2]. He also characterized those algebras of characteristic $p \neq 2, 3, 5$ that have at least one stable idempotent [3; 4]. Recently Kokoris announced [9] that every simple, flexible, power-associative algebra over an algebraically closed field of characteristic $\neq 2, 3$ that is of degree two and in which every idempotent is nilstable is a J -simple algebra.

It is the purpose of this paper to fill in the remaining gap by giving a characterization of those algebras $\mathfrak{A}$ that have an idempotent that is not nilstable. An example is also given of an algebra $\mathfrak{A}$ that does not have a stable idempotent.

1. Let $\mathfrak{A}$ be an algebra that is simple, commutative, power-associative, of degree two and whose base field $\mathfrak{F}$ is an algebraically closed field of characteristic $p \neq 2, 3, 5$. Let e be a primitive idempotent of $\mathfrak{A}$ that is not nilstable. Since $\mathfrak{A}$ is power-associative we have $x^2x^2 = x^4$ for all $x \in \mathfrak{A}$ and the linearization of this identity

$$\begin{aligned} P(x, y, s, t) = & 4(xy)(st) + 4(xs)(yt) + 4(xt)(ys) \\ (1) \quad & - x[y(st) + s(yt) + t(ys)] - y[x(ts) + t(xs) + s(xt)] \\ & - s[x(yt) + y(xt) + t(xy)] - t[x(ys) + y(xs) + s(xy)] = 0. \end{aligned}$$

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We will use $\mathfrak{C}$ to represent the space $\mathfrak{A}_e(1) + \mathfrak{A}_e(0)$, a_λ to represent the $\mathfrak{A}_e(\lambda)$ -component of a , a_{10} to represent the $\mathfrak{C}$ -component of a , and z to represent $e-f$. We will make frequent use of some of the results of Albert on commutative power-associative algebras; namely, results (5), (6), (7), (8) of [1]. We state them as

$$(2) \quad [g(xy)_\lambda]_{1/2} = [(gx_\lambda)_{1/2}y_\lambda]_{1/2} + [(gy_\lambda)_{1/2}x_\lambda]_{1/2},$$

$$(3) \quad [g(xy)_\lambda]_{1-\lambda} = 2[(gx_\lambda)_{1/2}y]_{1-\lambda} + 2[(gy_\lambda)_{1/2}x]_{1-\lambda},$$

$$(4) \quad [(gx_\lambda)_{1/2}y_{1-\lambda}]_{1/2} = [(gy_{1-\lambda})_{1/2}x_\lambda]_{1/2},$$

$$(5) \quad (gx_\lambda)_{1-\lambda}y_{1-\lambda} = 2[(gy_{1-\lambda})_{1/2}x_\lambda]_{1-2},$$

where $\lambda = 1, 0; g \in \mathfrak{A}_e(1/2)$ and x and y are in $\mathfrak{C}$.

Two other relations

$$(6) \quad 2[(x_\lambda g)_{1/2}g]_\lambda + [(x_\lambda g)_{1-\lambda}g]_\lambda = x_\lambda g^2,$$

$$(7) \quad (x_1 g)_{1/2} = (x_0 g)_{1/2} \text{ implies } (x_1^2 g)_{1/2} = (x_0^2 g)_{1/2}$$

for x and g as above will be useful. The first of these is obtained from $P(x, e, g, g) = 0$ while the second can be derived from (2) and (4).

THEOREM 1. $\mathfrak{C}$ is an associative subalgebra of $\mathfrak{A}$ with an element $c \in \mathfrak{C}$ such that there is a $w \in \mathfrak{A}_e(1/2)$ with $z(cw) = 1$, $(c_1 w)_{1/2} = (c_0 w)_{1/2}$ and $(c_1^2 w)_0 = -2c_0$.

Proof. It is easily seen that the subset $\mathfrak{I}$ of $\mathfrak{A}_e(1)$ consisting of all elements of the form $(a_0 g)_1$ is an ideal of $\mathfrak{A}_e(1)$ where $g \in \mathfrak{A}_e(1/2)$ and a_0 is a fixed element of $\mathfrak{A}_e(0)$ because by (5) we have $b_1(a_0 g)_1 = 2[a_0(b_1 g)_{1/2}]_1$. The additive property of an ideal is immediate.

We now let b_1, d_1 be elements of $\mathfrak{A}_e(1)$, $g \in \mathfrak{A}_e(1/2)$ and $a_0 \in \mathfrak{A}_e(0)$ with $(a_0 g)_1 = a_1$. If we consider only the $\mathfrak{A}_e(1)$ -components of each of the terms in $P(b_1, d_1, g, a_0) = 0$ we get $2(b_1 d_1)_1 a_1 = b_1(d_1 a_1) + d_1(b_1 a_1)$. If b_1 is also in $\mathfrak{I}$ we can interchange a_1 and b_1 to get $a_1(d_1 b_1) = 2b_1(d_1 a_1) - d_1(b_1 a_1)$. Therefore $a_1(d_1 b_1) = (a_1 d_1)b_1$. Hence $\mathfrak{I}$ is associative.

It has been shown [1, Lemma 11] that if $(a_0 g)_1 \in \mathfrak{N}_1$ for all $a_0 \in \mathfrak{A}_e(0)$ and $g \in \mathfrak{A}_e(1/2)$ then $(a_1 g)_0 \in \mathfrak{N}_0$ for all $a_1 \in \mathfrak{A}_e(1)$ and $g \in \mathfrak{A}_e(1/2)$. From this result and the assumption that e is not nilstable we can conclude that there is an element $c_0 \in \mathfrak{A}_e(0)$ and an element g in $\mathfrak{A}_e(1/2)$ such that $(c_0 g)_1$ is nonsingular. If b_1 is the inverse of $(c_0 g)_1$ in $\mathfrak{A}_e(1)$ then $[c_0(2b_1 g)_{1/2}]_1 = b_1(c_0 g)_1 = e$. We may also conclude that $\mathfrak{A}_e(1) = \mathfrak{I}$ is associative. In a similar manner we obtain the result that $\mathfrak{A}_e(0)$ is associative.

If we take $c_0 \in \mathfrak{A}_e(0)$ and $w \in \mathfrak{A}_e(1/2)$ such that $(c_0 w)_1 = e$ and let $2c_1 = (c_0^2 w)_1 = 4[c_0(c_0 w)_{1/2}]_1$ then we can quote the results of Kokoris [7, Lemma 4 and Identity 29] that $(c_1 w)_0 = -f$ or 0 and $(c_1 w)_{1/2} = (c_0 w)_{1/2}$. No generality will be lost if we also assume that c_0 is nilpotent because $\mathfrak{A}_e(0) = f\mathfrak{F} + \mathfrak{N}_0$ and

$(c_0w)_1 = [\alpha f + c_0w]_1$ for any $\alpha \in \mathfrak{F}$. To complete the proof of the theorem it remains only to show that $(c_1w)_0 \neq 0$. We assume that $(c_1w)_0 = 0$. If we examine the $\mathfrak{A}_e(1)$ -components of the terms of the relation $P(c_0, c_0, w, w) = 0$ we get $8(c_0w)_1^2 + 8[(c_0w)_{1/2}]_1 = 4[c_0[w(wc_0)_1]_{1/2}]_1 + 2[w(c_0^2w)_{1/2}]_1 + 4[w[c_0(c_0w)_{1/2}]_{1/2}]_1$. Using this relation together with (2), (6), (7) and $(c_0w)_1 = e$, we get $6e + 8[(c_1w)_{1/2}]_1 = 2w^2c_1^2 - 2[w(c_1^2w)_0]_1$. But $(c_1^2w)_0 = 4[c_1(c_1w)_{1/2}]_0 = 4[c_1(c_0w)_{1/2}]_0 = 2c_0(c_1w)_0 = 0$. Therefore either $[(c_1w)_{1/2}]_1$ or $w^2c_1^2$ must be nonsingular. If we again use (1) with $P(c_1, c_1, w, w) = 0$ and examine the $\mathfrak{A}_e(0)$ -components of the resulting terms we get $8[(c_0w)_{1/2}]_0^2 = 2c_0^2w^2$. But then $[(c_0w)_{1/2}]_0$ is nilpotent. Since $(c_0w)_{1/2}^2 = \alpha 1 + n$ where $n \in \mathfrak{N}_1 + \mathfrak{N}_0$ [1, Lemma 10] we must also have $[(c_1w)_{1/2}]_1$ nilpotent. Now by (6) we have $2[(c_0w)_{1/2}w]_1 = 2[(c_1w)_{1/2}w]_1 = -[(c_1w)_0w]_1 + c_1w^2 = c_1w^2$. But $2[(c_0w)_{1/2}w]_0 = -[(c_0w)_1w]_0 + c_0w^2 = c_0w^2$ is nilpotent. Therefore c_1w^2 and $c_1^2w^2$ are nilpotent. We have arrived at a contradiction. Hence $(c_1w)_0 = -f$ and the theorem is proved.

THEOREM 2. *There is an isomorphism T between $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ such that for $b_1 \in \mathfrak{A}_e(1)$, $T(b_1)$ is the unique element of $\mathfrak{A}_e(0)$ satisfying $(b_1w)_{1/2} = [T(b_1)w]_{1/2}$. The subset $\mathfrak{B}$ of $\mathfrak{C}$ of all elements of the form $b_1 + T(b_1)$ is an associative subalgebra of $\mathfrak{C}$ isomorphic to both $\mathfrak{A}_e(0)$ and $\mathfrak{A}_e(1)$.*

Proof. We use c_1, c_0 and w as in Theorem 1. If we consider only the $\mathfrak{A}_e(1/2)$ -components of the terms in $P(c_0, b_1, w, w) = 0$ we get $8[(c_0w)_{1/2}(b_1w)_0]_{1/2} + 4(b_1w)_{1/2} = 2[w\{b_1 + [(b_1w)_{1/2}c_0]_1\}]_{1/2} + 2\{w[(c_0w)_{1/2}b_1 + (b_1w)_0c_0]_{1/2} + 2\{c_0[w(wb_1)_0]_{1/2}\}_{1/2} + (b_1w)_{1/2}\}$. Using (5) and (2) on the terms $[(c_0w)_{1/2}b_1]_0$, $[(b_1w)_{1/2}c_0]_1$ and $\{c_0[w(wb_1)_0]_{1/2}\}_{1/2}$ this relation reduces to $[(c_0w)_{1/2}(b_1w)_0]_{1/2} = \{w[(c_0w)_{1/2}b_1]_0\}_{1/2}$. We now consider the $\mathfrak{A}_e(1/2)$ -component of each term in $P(c_1, b_1, w, w) = 0$. We have

$$\begin{aligned} & -4(b_1w)_{1/2} + 8[(c_1w)_{1/2}(b_1w)_0]_{1/2} \\ & = 2\{w[(c_1b_1)w + c_1(b_1w)_{1/2} + b_1(c_1w)_{1/2}]_0\}_{1/2} \\ & \quad - (b_1w)_{1/2} + 2\{c_1[w(wb_1)_0]_{1/2}\}_{1/2}. \end{aligned}$$

This relation together with (2) and (4) gives us $2[(c_1w)_{1/2}(b_1w)_0]_{1/2} = (b_1w)_{1/2} + \{w[(c_1b_1)w]_0\}_{1/2}$. But $[(c_0w)_{1/2}(b_1w)_0]_{1/2} = \{w[(c_0w)_{1/2}b_1]_0\}_{1/2}$ and $(c_0w)_{1/2} = (c_1w)_{1/2}$. Therefore $(b_1w)_{1/2} = (\{2[(c_1w)_{1/2}b_1]_0 - [(c_1b_1)w]_0\}w)_{1/2} = -2\{[(b_1w)_{1/2}c_1]_0w\}_{1/2}$. We can now define $T(b_1) = -2[(b_1w)_{1/2}c_1]_0$ to be the element b_0 in $\mathfrak{A}_e(0)$ such that $(b_1w)_{1/2} = (b_0w)_{1/2}$. To show that T is well-defined we assume $(a_0w)_{1/2} = 0$. We have $a_0 = -a_0(c_1w)_0 = -2[c_1(a_0w)_{1/2}]_0 = 0$ by (5). Therefore $(b_0w)_{1/2} = (b'_0w)_{1/2}$ implies $b_0 = b'_0$. Simply by changing the signs of c_1 and c_0 and interchanging 1 and 0 we can get a similar result for $\mathfrak{A}_e(0)$; i.e., for every $b_0 \in \mathfrak{A}_e(0)$ there is a unique $b_1 = 2\{[(b_0w)_{1/2}c_0]_1w\}_{1/2}$ such that $(b_0w)_{1/2} = (b_1w)_{1/2}$. Therefore T is onto $\mathfrak{A}_e(0)$ and is a 1-1 correspondence between $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$.

Now if a and b are elements of $\mathfrak{B}$ as defined in the theorem we have, with the help of (2) and (4), that

$$\begin{aligned} [w(a_1b_1)]_{1/2} &= [(wb_1)a_1 + (wa_1)b_1]_{1/2} \\ &= [(wb_0)a_1 + (wa_0)b_1]_{1/2} = [(wa_1)b_0 + (wb_1)a_0]_{1/2} \\ &= [(wa_0)b_0 + (wb_0)a_0]_{1/2} = [w(a_0b_0)]_{1/2}. \end{aligned}$$

Therefore $T(a_1b_1) = a_0b_0$ and $ab = a_1b_1 + a_0b_0 \in \mathfrak{B}$. Clearly $\mathfrak{B}$ is closed under addition and scalar multiplication.

Define $S(b) = be$ for every $b \in \mathfrak{B}$. It follows immediately from the above results that S is a 1-1 correspondence of $\mathfrak{B}$ onto $\mathfrak{U}_e(1)$. From the definition we have $S(ab) = (ab)e = (ae)(be) = S(a)S(b)$ and $S(a+b) = S(a) + S(b)$ for all a and b in $\mathfrak{B}$. Therefore $\mathfrak{B}$ and $\mathfrak{U}_e(1)$ are isomorphic as rings and hence as algebras. In the same manner we show that $\mathfrak{B}$ is isomorphic to $\mathfrak{U}_e(0)$. We have shown also that T is an isomorphism. The associativity of $\mathfrak{B}$ follows from that of $\mathfrak{C}$.

From the definition of $\mathfrak{B}$ it is clear that $c = c_1 + c_0$ is in $\mathfrak{B}$. From $P(w, w, w, z) = 0$ it follows that w^2 is in $\mathfrak{B}$. Theorem 2 also implies that $\mathfrak{C} = \mathfrak{B} + \mathfrak{B}z$.

THEOREM 3. *The mapping $b \rightarrow D(b) = (bw)z$ is a derivation of $\mathfrak{B}$ into $\mathfrak{B}$ such that $D(c) = 1$.*

Proof. Let a and b be arbitrary elements of $\mathfrak{B}$. Then $[(ab)w]_{10} = [(ab)_1w]_0 + [(ab)_0w]_1 = [(a_1b_1)w]_0 + [(a_0b_0)w]_1 = 2[a_1(b_1w)_{1/2}]_0 + 2[b_1(a_1w)_{1/2}]_0 + 2[a_0(b_0w)_{1/2}]_1 + 2[b_0(a_0w)_{1/2}]_1 = 2[a_1(b_0w)_{1/2}]_0 + 2[b_1(a_0w)_{1/2}]_0 + 2[a_0(b_1w)_{1/2}]_1 + 2[b_0(a_1w)_{1/2}]_1 = b_0(a_1w)_0 + a_0(b_1w)_0 + b_1(a_0w)_1 + a_1(b_0w)_1 = b(aw)_{10} + a(bw)_{10}$ by (3), (5) and the definition of $\mathfrak{B}$. If this relation is multiplied by z we have $D(ab) = aD(b) + bD(a)$ and D is a derivation on $\mathfrak{B}$ into $\mathfrak{C}$.

To show that $D(b)$ lies in $\mathfrak{B}$ for $b = b_1 + b_0$, an element of $\mathfrak{B}$, we need several identities; the first of which is obtained from $P(b_0, w, w, c_1) = 0$. We get $8[(b_0w)_1(wc_1)_{1/2}]_0 + 8[(b_0w)_{1/2}(wc_1)_{1/2}]_0 = 2(b_0c_0)w^2 + 2\{[c_1(wb_0)_1]w\}_0$ after the usual simplifications using (2), (5), (6) and $(c_1w)_{1/2} = (c_0w)_{1/2}$. $8[(b_0w)_{1/2}(wc_1)_{1/2}]_0 = 2(b_0c_0)w^2 + 2\{[c_1(wb_0)_1]w\}_0$ after the usual simplifications. We consider $P(b_0, w, w, c_0) = 0$ next to get

$$\begin{aligned} &-3(b_0c_0)w^2 + 8[(b_0w)_{1/2}(wc)_{1/2}]_0 + 6[(b_0w)_1(c_0w)_{1/2}]_0 \\ &= -2\{[(b_0c_0)w]_{1/2}w\}_0. \end{aligned}$$

Finally we obtain $3(b_1w)_0 = 4[(b_1w)_{1/2}(c_0w)_{1/2}]_0 - 2\{w[(c_0b_0)w]_{1/2}\}_0$ from $P(b_1, w, w, c_1) = 0$. Now, from the proof of Theorem 2 and from (3) we have

$$\begin{aligned} 6T[b_0w]_1 &= -12\{[(b_0w)_1w]_{1/2}c_1\}_0 \\ &= 12[(c_1w)_{1/2}(b_0w)_1]_0 - 6\{[(b_0w)_1c_1]w\}_0. \end{aligned}$$

By successively applying to this relation the three identities above in the order we obtained them we get $6T[(b_0w)_1] = -12[(b_0w)_1(c_1w)_{1/2}]_0 - 24[(b_0w)_{1/2}(c_1w)_{1/2}]_0 + 6b_0c_0w^2 = -12[(b_0w)_1(c_0w)_{1/2}]_0 - 24[(b_0w)_{1/2}(c_0w)_{1/2}]_0 + 6c_0b_0w^2 = 4\{[(b_0c_0)w]_{1/2}w\}_0 - 8[(b_0w)_{1/2}(c_0w)_{1/2}]_0 = -6(b_1w)_0$. Therefore we have $D(b)z = (bw)_{10}z = (b_0w)_1 - (b_1w)_0 \in \mathfrak{B}$. The fact that $D(c) = 1$ follows immediately from the definition of c .

THEOREM 4. *If a and b are elements of $\mathfrak{B}$ then $[(wa)_{1/2}b]_{1/2} = [w(ab)]_{1/2}$, $[(wa)_{1/2}b]_{10} = (wb)_{10}a$ and $(wa)_{1/2}(wb)_{1/2} \in \mathfrak{B}$.*

Proof. By (2) and (4) and the definition of $\mathfrak{B}$ we have $[w(ab)]_{1/2} = 2[w(ab)_1]_{1/2} = 2[(wa)_1]_{1/2}b_1 + 2[(wb)_1]_{1/2}a_1 = [(wa)_{1/2}b_1]_{1/2} + 2[(wb)_1]_{1/2}a_1 = [(wa)_{1/2}b_1]_{1/2} + 2[(wa)_1]_{1/2}b_0 = [(wa)_{1/2}b]_{1/2}$. By (5) we have $[(wa)_{1/2}b]_{10} = 2[(wa)_1]_{1/2}b_1 + 2[(wa)_1]_{1/2}b_0 = (wb)_1a_0 + (wb)_1a_1 = (wb)_{10}a$. Now use $P(w, w, a, b)$ to get $4w^2ab + 8(wa)(wb) = 2w[(ab)w + (aw)b + (bw)a] + a[w^2b + 2w(wb)] + b[w^2a + 2w(wa)]$. If we consider only the $\mathbb{C}$ -components of each of the terms and if we use the facts that $(wa)_{10} \in \mathfrak{B}z$ for all $a \in \mathfrak{B}$ and $[w(az)]_{10} \in \mathfrak{B}$ then $8(wa)_{1/2}(wb)_{1/2} - 4w[(ab)w]_{1/2} - 2a[w(wb)_{1/2}] - 2b[w(wa)_{1/2}]$ is in $\mathfrak{B}$. Now $P(w, w, az, z) = 0$ implies $2w^2a = -2D^2(a) + 2w(wa)_{1/2}$. Since a, w^2 and $D(a)$ are in $\mathfrak{B}$, so also is $w(wa)_{1/2}$. Hence $8(wa)_{1/2}(wb)_{1/2}$ is in $\mathfrak{B}$.

COROLLARY. *If $a \in \mathbb{C}$ and $b \in \mathfrak{B}$ then $[(wa)_{1/2}b]_{1/2} = [w(ab)]_{1/2}$.*

Proof. We can write $a = a' + a''z$ where a' and a'' are in $\mathfrak{B}$. Since $[(a''z)w]_{1/2} = [(a''bz)w]_{1/2} = 0$ we have $[(wa)_{1/2}b]_{1/2} = [(wa')_{1/2}b]_{1/2} = [w(a'b)]_{1/2} = [w(ab)]_{1/2}$.

We now define $\mathfrak{G}$ to be the set of all $g \in \mathfrak{U}_e(1/2)$ such that $(gc)_{10}$ is in $\mathfrak{B}$.

THEOREM 5. $\mathfrak{U}_e(1/2)$ is the direct sum of the two subspaces $(w\mathfrak{B})_{1/2}$ and $\mathfrak{G}$. Moreover $(\mathfrak{G}a)_{1/2} \subseteq \mathfrak{G}$, $[\mathfrak{G}(az)]_{1/2} \subseteq (w\mathfrak{B})_{1/2}$, and $[(w\mathfrak{B})_{1/2}(az)]_{1/2} \subseteq \mathfrak{G}$, for all $a \in \mathfrak{B}$.

Proof. If g is any element of $\mathfrak{U}_e(1/2)$, let $(gc)_{10} = a + a'z$ where a and a' are in $\mathfrak{B}$. Since $[(a'w)_{1/2}c]_{10} = a'z$ we have $\{[g - (a'w)_{1/2}c]\}_{10} = a, [g - (a'w)_{1/2}c] \in \mathfrak{G}$ and g is equal to the sum of an element of $\mathfrak{G}$ and an element of $(w\mathfrak{B})_{1/2}$. If h lies in both $(w\mathfrak{B})_{1/2}$ and $\mathfrak{G}$ then $(hc)_{10}$ lies in $\mathfrak{B}z$ and $\mathfrak{B}$. Hence $(hc)_{10} = 0$. But $[(wa)_{1/2}c]_{10} = az$. Therefore if $h = (wa)_{1/2}$ then $a = (wa)_{1/2} = 0$ and $h = 0$. Hence $\mathfrak{U}_e(1/2)$ is the direct sum of $\mathfrak{G}$ and $(w\mathfrak{B})_{1/2}$.

Since $D(c^2) = 2c$, the $\mathfrak{U}_e(1/2)$ -components of the terms obtained from $P(c, c, w, g) = 0$ with $g \in \mathfrak{G}$ yield the relation

$$\begin{aligned} 8[(cw)_{1/2}(cg)_{10}]_{1/2} &= \{2c[w(cg)_{10}]_{1/2} + w(c^2g)_{10} + 2w[c(cg)_{10}]\}_{1/2} \\ &\quad + \{2w[c(cg)_{1/2}]_{10} + 6g(cz)\}_{1/2}. \end{aligned}$$

Using this relation, Theorem 4 and the property that $(cg)_{10} \in \mathfrak{B}$ it is easily seen that $[g(cz)]_{1/2}$ is in $(w\mathfrak{B})_{1/2}$. Therefore $[g(cz)_{1/2}c]_{10} \in z\mathfrak{B}$. But

$$\begin{aligned} [g(cz)_{1/2}c]_{10} &= [(gc_1)_{1/2}c_1 - (gc_0)_{1/2}c_1 - (gc_0)_{1/2}c_0 + (gc_1)_{1/2}c_0]_{10} \\ &= [(1/4)(c_1^2g) - (1/4)(c_0^2g) - (1/2)(gc_1)_0c_0 + (1/2)(gc_0)_1c_1]_{10} \\ &= -(1/4)(c^2g)_{10}z + (1/2)(cz)(cg)_{10}. \end{aligned}$$

Therefore since $(cg)_{10}$ is an element of $\mathfrak{B}$ we also have $(c^2g)_{10}$ is an element of $\mathfrak{B}$. Similarly $[(cg)_{1/2}c]_{10} = (1/4)(c^2g)_{10} + (1/2)c(cg)_{10}$ is in $\mathfrak{B}$. Therefore $(cg)_{1/2}$ is in $\mathfrak{G}$. We now examine the $\mathfrak{U}_e(1/2)$ -components of the terms resulting from $P(a_1, c_1, w, g) = 0$. With the help of (3) and (4) we get

$$\begin{aligned} (8) \quad & [2(a_1w)_0(c_1g)_{1/2} + 2(a_1w)_{1/2}(c_1g)_0 + 2(c_1w)_{1/2}(a_1g)_0 + 2(c_1w)_0(a_1g)_{1/2}]_{1/2} \\ &= \{w[(a_1c_1)g]_0 + g[(a_1c_1)w]_0\}_{1/2}. \end{aligned}$$

Interchanging the subscripts 1 and 0 we obtain

$$\begin{aligned} & [2(a_0w)_1(c_0g)_{1/2} + 2(a_0w)_{1/2}(c_0g)_1 + 2(c_0w)_{1/2}(a_0g)_1 + 2(c_0w)_1(a_0g)_{1/2}]_{1/2} \\ &= \{w[(a_0c_0)g]_1 + g[(a_0c_0)w]_1\}_{1/2}. \end{aligned}$$

But

$$\begin{aligned} \{g[(a_0c_0)w]_1\}_{1/2} &= \{2g[a_0(c_0w)_{1/2} + c_0(a_0w)_{1/2}]_1\}_{1/2} \\ &= \{2g[a_0(c_1w)_{1/2}]_1 + g[a_1(c_0w)_1]_1\}_{1/2} \\ &= \{g[c_1(a_0w)_1] + ga_1\}_{1/2}. \end{aligned}$$

Therefore

$$\begin{aligned} (9) \quad & [2(a_0w)_1(c_0g)_{1/2} + 2(a_0w)_{1/2}(c_0g)_1 + 2(c_0w)_{1/2}(a_0g)_1 + 2(c_0w)_1(a_0g)_{1/2}]_{1/2} \\ &= \{g[c_1(a_0w)_1] + w[(a_0c_0)g]_1 + [a_1g]\}_{1/2}. \end{aligned}$$

Again consider only the $\mathfrak{U}_e(1/2)$ -components of the terms of $P(a_0, c_1, w, g) = 0$. This relation together with (2), (3) and (4) gives us

$$\begin{aligned} (10) \quad & [2(a_0w)_{1/2}(c_1g)_0 + 2(a_0w)_1(c_1g)_{1/2} + 2(a_0g)_1(c_1w)_{1/2} + 2(c_1w)_0(a_0g)_{1/2}]_{1/2} \\ &= \{[a_0(c_1w)_0]g + g[c_1(a_0w)_1] + w[a_0(c_1g)_0] + w[c_1(a_0g)_1]\}_{1/2}. \end{aligned}$$

Interchanging the subscripts 0 and 1 in (10) we obtain

$$\begin{aligned} (11) \quad & [2(a_1w)_{1/2}(c_0g)_1 + 2(a_1w)_0(c_0g)_{1/2} + 2(a_1g)_0(c_0w)_{1/2} + 2(c_0w)_1(a_1g)_{1/2}]_{1/2} \\ &= \{[a_1(c_0w)_1]g + g[c_0(a_1w)_0] + w[a_1(c_1g)_1] + w[c_0(a_1g)_0]\}_{1/2}. \end{aligned}$$

We now subtract the sum of identities (10) and (11) from the sum of the identities (8) and (9) and use the facts that $(a_1w)_{1/2} = (a_0w)_{1/2}$, $(c_1w)_0 = -f$ and $(c_0w)_1 = e$. We have $\{2[(az)w]_{10}[(cz)g]_{1/2} + (a_0g) - 2(a_1g) - g[(a_1c_1)w]_0 + g[c_0(a_1w)_0]\}_{1/2}$ is in $(w\mathfrak{B})_{1/2}$. Therefore

$$\begin{aligned}
& \{-2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g) - 2g[(a_1c_1w)_{1/2}]_0 \\
& \quad - 2g[c_1(a_1w)_{1/2}]_0 + 2g[a_1(c_0w)_{1/2}]_0\}_{1/2} \\
& = \{-2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g) - 2g[a_1(c_1w)_{1/2}]_0 \\
& \quad - g[a_0(c_1w)_0] + 2g[a_1(c_1w)_{1/2}]_0\}_{1/2} \\
& = \{-2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g) + ga_0\}_{1/2} \\
& = \{-2D(a)[(cz)g]_{1/2} - 2(az)g\}_{1/2}
\end{aligned}$$

is in $\mathfrak{B}$. Since $[(cz)g]_{1/2} \in (w\mathfrak{B})_{1/2}$ we have $[(az)g]_{1/2} \in (w\mathfrak{B})_{1/2}$. To show that $(ga)_{1/2} \in \mathfrak{G}$ for $a \in \mathfrak{B}$ we consider

$$\begin{aligned}
[(ga)_{1/2}c]_{10} &= [(ga_1)_{1/2}c_1 + (ga_0)_{1/2}c_1 + (ga_1)_{1/2}c_0 + (ga_0)_{1/2}c_0]_{10} \\
&= [2(ga_0)_{1/2}c_1 + [g(az)]_{1/2}c_1 + 2(ga_1)_{1/2}c_0 - [g(az)]_{1/2}c_0]_{10} \\
&= [(gc_{10}a_0 + (gc_0)_1a_1 + g(az)_{1/2}(cz)]_{10} \\
&= (gc)_{10}a + \{[g(az)]_{1/2}(cz)\}_{10}.
\end{aligned}$$

Since $(gc)_{10} \in \mathfrak{B}$ so is $(gc)_{10}a$. Also since $[g(az)]_{1/2} \in (w\mathfrak{B})_{1/2}$ we have

$$\{[g(az)]_{1/2}(cz)\}_{10} \in \mathfrak{B}.$$

Hence $[(ga)_{1/2}c]_{10} \in \mathfrak{B}$ and $(ga)_{1/2} \in \mathfrak{G}$. Finally if we take a, b and h in $\mathfrak{B}$ we have $\{[(wa_0)_{1/2}(bz)]_{1/2}h_1\}_0 = \{[(wa_0)_{1/2}b_1]_{1/2}h_1 - [(wa_0)_{1/2}b_0]_{1/2}h_1\}_0 = \{[(wb_1)a_0]_{1/2}h_1 - (1/4)(wh_1)_0a_0b_0\}_0 = \{[(wb_0)a_0]_{1/2}h_1 - (1/4)(wh_1)_0a_0b_0\}_0 = (1/4)(wh_1)_0b_0a_0 - (1/4)(wh_1)_0b_0a_0 = 0$. Similarly $\{[(wa_0)_{1/2}(bz)]_{1/2}h_0\}_1 = 0$. By taking $h = c$ we can see that the $(w\mathfrak{B})_{1/2}$ component of $[(wa)_{1/2}(bz)]_{1/2}$ is 0. Hence $[(wa)_{1/2}(bz)]_{1/2}$ is in $\mathfrak{G}$.

THEOREM 6. $[(w\mathfrak{B})_{1/2}(\mathfrak{B}z)]_{1/2} = 0$.

Proof. Let a be a nilpotent element of $\mathfrak{A}_e(1)$. There exists a $\lambda \in \mathfrak{F}$ such that $d = a + \lambda c$ has the property that $(d_0w)_1$ is a nonsingular element b_1 of $\mathfrak{A}_e(1)$. Then $[d(2b_1^{-1}w)_{1/2}]_1 = b_1^{-1}(dw)_1 = e$. If we let b be the unique element of $\mathfrak{B}$ whose $\mathfrak{A}_e(1)$ -component is b_1 we have by the isomorphism established in Theorem 2 that $[d(b^{-1}w)_{1/2}]_{10} = b^{-1}D(d)z = z$. For these elements $d \in \mathfrak{C}$ and $(wb^{-1})_{1/2} \in \mathfrak{A}_e(1/2)$ we get a $\mathfrak{B} \subseteq \mathfrak{C}$ such that $\mathfrak{B} + \mathfrak{B}z = \mathfrak{C}$ and where $\mathfrak{B}$ has the properties described for $\mathfrak{B}$ in Theorems 2-5. Let $t + sz \in \mathfrak{B}z$ where t and $s \in \mathfrak{B}$. We have $[(wb^{-1})_{1/2}(t + sz)]_{1/2} = 0$. Therefore $(wb^{-1}t)_{1/2} + [(wb^{-1})_{1/2}(sz)]_{1/2} = 0$. Since $[(wb^{-1})_{1/2}(sz)]_{1/2} \in \mathfrak{G}$ we must have $(wb^{-1}t)_{1/2} = 0$ and $b^{-1}t = 0$. Therefore $t = 0$ and $\mathfrak{B}z \subseteq \mathfrak{B}z$. If $\mathfrak{B}z$ is a proper subset of $\mathfrak{B}z$ then $\mathfrak{B}$ is a proper subset of $\mathfrak{B}$. But this would imply that $\mathfrak{B} + \mathfrak{B}z$ is a proper subset of $\mathfrak{C}$ which is a contradiction. Therefore we must have $\mathfrak{B}z = \mathfrak{B}z$ and $[(wb^{-1})_{1/2}(\mathfrak{B}z)]_{1/2} = 0$. Now let $\mathfrak{S}$ be the subset of $\mathfrak{B}$ of all elements s such that $[(ws)_{1/2}(\mathfrak{B}z)]_{1/2} = 0$. Let $x, y \in \mathfrak{B}$. The re-

lation $P(y, x, w, z) = 0$ yields $[(wx)(yz)]_{1/2} + [(wy)(xz)]_{1/2} = 0$. Let $t \in \mathfrak{B}$, s and $s' \in \mathfrak{S}$. Then we get $\{[w(ss')]_{1/2}(tz)\}_{1/2} = -\{(wt)_{1/2}(ss'z)\} = 0$ from $P(tw, s, s'z) = 0$. Hence $\mathfrak{S}$ is a subalgebra of $\mathfrak{B}$. If we let $b_{-1} = \alpha + n$ where b is as described above and n is a nilpotent element of $\mathfrak{B}$ and $\alpha \in \mathfrak{F}$, then $n \in \mathfrak{S}$ and hence every power of n is in $\mathfrak{S}$. But b is the sum of a multiple of the identity and a linear combination of powers of n . Hence $b = \lambda + D(a) \in \mathfrak{S}$ and the derivative of every element of $\mathfrak{B}$ is in $\mathfrak{S}$. Now $a \in \mathfrak{B}$ implies $a = D(ca) - cD(a)$. Since $D(ca)$, $c = (1/2)D(c^2)$ and $D(a)$ are in $\mathfrak{S}$ we have $\mathfrak{B} \subseteq \mathfrak{S}$ and $[(w\mathfrak{B})_{1/2}(\mathfrak{B}z)]_{1/2} = 0$.

At this point we have obtained partial results on the multiplications of $\mathfrak{A}$. However, the chief remaining gap in the characterization of $\mathfrak{A}$ lies with the products involving elements of $\mathfrak{G}$. To facilitate the determination of these products we shall introduce some symbols Q_g , ϕ_g , k_g , f_g , and h_g on $\mathfrak{B}$ into $\mathfrak{B}$ for every $g \in \mathfrak{G}$ by letting

$$(12) \quad [g(bz)]_{1/2} = [wQ_g(b)]_{1/2},$$

$$(13) \quad (gb)_{10} = h_g(b) + k_g(b)z,$$

$$(14) \quad [g(wb)_{1/2}]_{10} = f_g(b) + \phi_g(b)z$$

for every $b \in \mathfrak{B}$. In our subscripts we abbreviate $(ga)_{1/2}$ to ga .

From (2) and (3) and the definition of $\mathfrak{G}$ we have

$$\begin{aligned} \{[(ga)_{1/2}(bz)]_{1/2}c\}_1 &= \{[(ga)_{1/2}b_1]_{1/2}c_0\}_1 - \{[(ga)_{1/2}b_0]_{1/2}c_0\}_1 \\ &= \{2[(ga_1)_{1/2}b_1]_{1/2}c_0 + [(wQ_g(a))_{1/2}b_1]_{1/2}c_0 \\ &\quad - 2[(ga_1)_{1/2}b_0]_{1/2}c_0 - [(wQ_g(a))_{1/2}b_1]_{1/2}c_0\}_1 \\ &= (1/2)(gc_0)_1 a_1 b_1 + (1/2)b_1 Q_g(a) - (1/2)b_1 Q_g(a) \\ &\quad - 2\{[(gb_0)_{1/2}a_1]_{1/2}c_0\}_1 \\ &= (1/2)(gc_0)_1 a_1 b_1 - 2\{[(gb_1)_{1/2}a_1]_{1/2}c_0\}_1 \\ &\quad + 2\{[(wQ_g(b))_{1/2}a_1]_{1/2}c_0\}_1 \\ &= a_1 Q_g(b). \end{aligned}$$

Now $[(ga)_{1/2}(bz)]_{1/2} = [wQ_{ga}(b)]_{1/2}$ and therefore $[(wQ_{ga}(b))_{1/2}c]_{10} = Q_{ga}(b)z$.

Hence

$$(15) \quad Q_{ga}(b) = aQ_g(b).$$

Consider $h_{gb}(a) + k_{gb}(a)z = [(gb)_{1/2}a]_{10} = [(gb)_{1/2}a_1]_0 + (gb)_{1/2}a_0]_1$
 $= 2[(gb_0)_{1/2}a_1]_0 + [(wQ_g(b))_{1/2}a_1]_0 + 2[(gb_1)_{1/2}a_0]_1 - [(wQ_g(b))_{1/2}a_0]_1$
 $= b_0(ga_1)_0 + b_1(ga_0)_1 + Q_g(b)[(az)w]_{10} = bh_g(a) + bz k_g(a) - Q_g(b)D(a)$.
 From this relation we obtain

$$(16) \quad h_{gb}(a) = bh_g(a) - Q_g(b)D(a),$$

$$(17) \quad k_{gb}(a) = bk_g(a).$$

We now consider the $\mathfrak{U}$ -components of the terms of $P(a, a, g, z) = 0$. We have $3ah_g(a)z + 3ak_g(a) - 5h_{ga}(a)z - 5k_{ga}(a) = Q_g(a)D(a)z - h_g(a^2)z - k_g(a^2)$. If we equate $\mathfrak{B}$ -components and $\mathfrak{B}z$ -components we have

$$(18) \quad k_g(a^2) = 2ak_g(a),$$

$$(19) \quad h_g(a^2) = 2ah_g(a) - 4Q_g(a)D(a)$$

by using (16) and (17).

We have proved that k_g is a derivation for every $g \in \mathfrak{G}$. We shall now prove that Q_g is a derivation for every $g \in \mathfrak{G}$. We have

$$\begin{aligned} [wQ_g(ab)]_{1/2} &= [g(abz)]_{1/2} = [g(ab)_1]_{1/2} - [g(ab)_0]_{1/2} \\ &= [(ga_1)_{1/2}b_1 + (gb_1)_{1/2}a_1 - (ga_0)_{1/2}b_0 - (gb_0)_{1/2}a_0]_{1/2} \\ &= [(ga_1)_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1 - (ga_0)_{1/2}b_0 \\ &\quad - (gb_1)_{1/2}a_0 + (wQ_g(b))_{1/2}a_0]_{1/2} \\ &= [(ga_0)_{1/2}b_1 + (wQ_g(a))_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1 \\ &\quad - (ga_1)_{1/2}b_0 + (wQ_g(a))_{1/2}b_0 - (gb_1)_{1/2}a_0 + (wQ_g(b))_{1/2}a_0]_{1/2} \\ &= [(ga_0)_{1/2}b_1 - (gb_1)_{1/2}a_0 + (gb_0)_{1/2}a_1 - (ga_1)_{1/2}b_0 \\ &\quad + w(Q_g(a)b) + w(Q_g(b)a)]_{1/2}. \end{aligned}$$

By (4) we have $(wQ_g(ab))_{1/2} = [w(Q_g(a)b + Q_g(b)a)]_{1/2}$. Therefore

$$(20) \quad Q_g(ab) = Q_g(a)b + Q_g(b)a.$$

Next, we consider the $\mathfrak{G}$ -components of the terms of $P(g, a, bz, z) = 0$ to get $4[(ga)b]_{1/2} = [3g(ab) + (gb)a]_{1/2}$. However

$$\begin{aligned} [(ga)b]_{1/2} &= [2(ga_0)b_1 + (wQ_g(a))b_1 + 2(ga_1)b_0 - (wQ_g(a))b_0]_{1/2} \\ &= 2[(gb_1)a_0 + (gb_0)a_1]_{1/2} \\ &= [(gb)a_0 + (wQ_g(b))a_0 + (gb)a_1 - (wQ_g(b))a_1]_{1/2} \\ &= [(gb)a]_{1/2}. \end{aligned}$$

If we combine the above two relations we have

$$(21) \quad [(ga)b]_{1/2} = [g(ab)]_{1/2}.$$

A similar computation using $P(w, w, a, z) = 0$ and $P((wa)_{1/2}, w, a, z) = 0$ gives us

$$(22) \quad w(wa)_{1/2} = w^2a + D^2(a)$$

$$(23) \quad (wa)_{1/2}^2 = w^2a^2 + 2aD^2(a) - D(a)D(a).$$

If we consider the $(w\mathfrak{B})_{1/2}$ -components of the terms of $P(z, (aw)_{1/2}, w, g) = 0$ we have $[wQ_g(w^2a) + wQ_g(D^2(a)) + w(a\phi_g(1)) + w\phi_g(a)]_{1/2} = 0$. By letting $a = 1$ we get

$$(24) \quad \phi_g(1) = -\frac{1}{2} Q_g(w^2).$$

Therefore

$$(25) \quad \phi_g(a) = \frac{1}{2} a Q_g(w^2) - Q_g(aw^2) - Q_g(D^2(a)).$$

From (15) and (25) we have

$$(26) \quad \phi_{ga}(b) = a\phi_g(b).$$

We now wish to express h_g in terms of Q_g and D . We examine the $\mathfrak{B}z$ -components of $P(w, g, c, a) = 0$ and use (21) and (26) to get $3\phi_g(c)a + 3\phi_g(a)c + 3h_g(a) + 3aD(h_g(c)) + 3D(a)h_g(c) + 3cD(h_g(a)) - 4h_{gc}(D(a)) = 3\phi_g(1)ca + D(h_g(ca)) + h_{gc}(a) + h_g(c)a + h_{ga}(c) + h_g(a)c + 3\phi_g(ca) - 3h_g(D(ca)) - ch_g(D(a)) + \phi_g(D(a))$. We simplify this relation using (25), (16) and the linearized form of (19) to get $-3Q_g(D^2(a))c + 3h_g(a) = -3Q_g(D^2(ca)) - Q_g(c)D^2(a) - 3D(Q_g(c))D(a) - 3D(Q_g(a)) - 3h_g(c)D(a) + 7Q_g(D(a))$. Since Q_g and D are derivations we have

$$(27) \quad 3h_g(a) = -3D(Q_g(c))D(a) - 3D(Q_g(a)) - 3h_g(c)D(a) + Q_g(D(a)) - 4Q_g(c)D^2(a).$$

If we let $a = c$ in (27) we get $h_g(c) = -D(Q_g(c))$. Therefore (27) simplifies to

$$(28) \quad 3h_g(a) = -3D(Q_g(a)) + Q_g(D(a)) - 4Q_g(c)D^2(a).$$

We substitute the values obtained from (28) in $h_g(ac) = ch_g(a) + ah_g(c) - 2Q_g(a) - 2Q_g(c)D(a)$, a linearized form of (19), to get

$$(29) \quad Q_g(a) = Q_g(c)D(a).$$

If we use this relation in (28) we obtain

$$(30) \quad h_g(a) = -D(Q_g(c))D(a) - 2Q_g(c)D^2(a).$$

We now investigate the behaviour of f_g . Consider the $\mathfrak{B}z$ -components of the terms of $P((wb)_{1/2}, g, a, z) = 0$. We have

$$(31) \quad 2f_g(b)a = f_{ga}(b) + f_g(ab) - bD(k_g(a)) - bk_g(D(a)) - D(a)k_g(b)$$

and when $b = 1$

$$(32) \quad 2f_g(1)a = f_{ga}(1) + f_g(a) - D(k_g(a)) - k_g(D(a)).$$

We define a new mapping T_g on $\mathfrak{B}$ into $\mathfrak{B}$ for each g by

$$(33) \quad T_g(a) = f_g(1)a - f_{ga}(1) + D(k_g(a)).$$

This definition together with (32) gives us $f_g(a) = f_g(1)a + T_g(a) + k_g(D(a))$ and $f_{ga}(1) = f_g(1)a - T_g(a) + D(k_g(a))$. Now $f_{ga}(b) = -f_g(ab) + 2f_g(b)a + b(Dk_g + k_gD)(a) + k_g(b)D(a)$ and $f_{ga}(b) = -f_{gab}(1) + 2f_{ga}(1)b + a(Dk_g + k_gD)(b) + D(a)k_g(b)$ by (31) and (32). Substituting the values for $f_g(ab)$, $f_g(b)$, $f_{gab}(1)$ and $f_{ga}(1)$ expressed in terms of T_g in these relations and simplifying we have

$$(34) \quad T_g(ab) = T_g(a)b + T_g(b)a$$

and

$$(35) \quad f_{ga}(b) = f_g(1)ab + T_g(b)a - bT_g(a) + ak_g(D(b)) + bD(k_g(a)) - k_g(a)D(b).$$

It follows readily that

$$(36) \quad T_{ga}(b) = aT_g(b) - D(b)k_g(a).$$

We have already shown that $\phi_g(a) = Q_g(c)[(1/2)aD(w^2) - D(w^2a) - D^3(a)]$. We also have that $P(g, g, (aw)_{1/2}, z) = 0$ implies $[g\phi_g(a)]_{1/2} = 0$. If we let $a = c^3$ we have $\phi_g(c^3) = Q_g(c)[-(1/2)c^3D(w^2) - 3c^2D(w^2) - 6]$. Since the second factor on the right-hand side is nonsingular we have $[gQ_g(c)]_{1/2} = 0$. Multiplying by cz and considering the $(w\mathfrak{B})_{1/2}$ -component we get

$$(37) \quad Q_g(c)^2 = 0.$$

Similarly we have

$$(39) \quad Q_g(c)k_g(a) = 0.$$

Now consider the element $w' = [w - wD(Q_g(c))]_{1/2} + g$ of $\mathfrak{A}_e(1/2)$. We have $(c_2w')_0 = -f$. By Theorem 1 and its proof, $c_2 - (1/2)(c_2^2w')_0$ is an element a in $\mathfrak{C}$ such that $(aw')z = 1$. Also $(c_2^2w')_0 = -2c_0 - 4(Q_g(c))_0$. Therefore $(aw')z = \{[c + Q_g(c) - Q_g(c)z]w'\}z = 1 - 2D(Q_g(c))^2 - 2D(Q_g(c))^2z - 2Q_g(c)D^2(Q_g(c))z + k_g(Q_g(c)) - 2Q_g(c)D^2(Q_g(c)) + k_g(Q_g(c))z$. Simple properties of derivations and the fact that $Q_g(c)^2 = 0$ gives us $(aw')z = 1 + k_g(Q_g(c)) + k_g(Q_g(c))z$. Therefore

$$(40) \quad k_g(Q_g(c)) = 0.$$

We also have from (35) and (36) that

$$(41) \quad T_g(Q_g(c)) = f_g(1)Q_g(c) \text{ and } T_g(b)Q_g(c) = 0$$

for every $b \in \mathfrak{B}$.

For w' and $c' = c + Q_g(c) - Q_g(c)z$ we have a corresponding $\mathfrak{B}'$ and $\mathfrak{B}'z$ as described in Theorem 2. To determine these two subspaces we let $a + bz$ be an element of $\mathfrak{C}$ with $a, b \in \mathfrak{B}$ and such that the $1/2$ -component of $w'(a + bz)$ is 0. We obtain $wa - wD(Q_g(c)a + ga + wQ_g(b))_{1/2} = 0$. Therefore $a[1 - D(Q_g(c))] = -Q_g(c)D(b)$. Solving for a we have $a = -D(b)Q_g(c)$. Since $\mathfrak{B}' + \mathfrak{B}'z = \mathfrak{C}$, we can conclude from the above result that $\mathfrak{B}'$ consists of all elements of the form $a - Q_g(a)z$. We note that the $\mathfrak{C}$ -component of the element $(a - Q_g(a)z)w'$ must be an element of $\mathfrak{B}'z$ by Theorem 3. If we calculate this element we obtain $D(a)z - D(a)D(Q_g(c))z + Q_g(c)D^2(a) + k_g(a)z - D(Q_g(a))D(Q_g(c)) + D(Q_g(c))^2 \cdot D(a)z$. In order for this element to be in $\mathfrak{B}'z$ we must have $Q_g(c)D^2(a) + D(Q_g(c))^2D(a) = Q_g(c)D[D(a) - D(a)D(Q_g(c)) + k_g(a) + D(Q_g(c))^2D(a)]$ by the definition of $\mathfrak{B}'z$. Therefore

$$(42) \quad Q_g(c)D(k_g(a)) = k_g(a)D(Q_g(c)) = 0.$$

We also have

$$(43) \quad Q_g(c)k_t(b) = 0$$

for any $t \in \mathfrak{G}$ and any $b \in \mathfrak{B}$ since $Q_g(c)k_t(b) = k_t(Q_g(c)b) - k_t(Q_g(c))b = k_t(Q_{gb}(c)) - k_t(Q_g(c))b = -k_{gb}(Q_t(c)) - k_t(Q_g(c))b = -bk_g(Q_t(c)) - k_t(Q_g(c))b = 0$.

We define t' to be the $1/2$ -component of

$$w[-D(Q_t(c)D(Q_g(c)) + Q_t(c)D^2(Q_g(c)) - k_t(Q_g(c)))] + t$$

for $t \in \mathfrak{G}$. Then the $\mathfrak{C}$ -component of $(c + Q_g(c) - Q_g(c)z)t'$ is

$$(44) \quad -D(Q_t(c)) - D(Q_t(c))D(Q_g(c)) - 2Q_t(c)D^2(Q_g(c)) + k_t(Q_g(c)) + Q_g(c)D^2(Q_t(c))z$$

since $Q_t(c)D^2(Q_g(c)) + 2D(Q_g(c))D(Q_t(c)) + Q_g(c)D^2(Q_t(c)) = 0$ and

$$2D(Q_t(c))D(Q_g(c))D(Q_g(c)) = -Q_t(c)D^2(Q_g(c))D(Q_g(c)) = 3Q_t(c)Q_g(c)D^3(Q_g(c)) = 0.$$

Hence t' is in $\mathfrak{G}'$. We now compute D' and Q'_t . We have simply that

$$(45) \quad D' : a - Q_g(a)z \rightarrow D(a) - D(Q_g(c))D(a) + D(Q_g(c))^2D(a) + k_g(a) \\ - [Q_g(c)D^2(a) + D(Q_g(c))^2D(a)]z,$$

$$(46) \quad Q'_t : c + Q_g(c) - Q_g(c)z \rightarrow Q_t(c) + Q_t(c)D(Q_g(c)) - Q_g(c)D(Q_t(c))z.$$

Therefore

$$D'Q'_t : c + Q_g(c) - Q_g(c)z \rightarrow D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) \\ + Q_t(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) + k_g(Q_t(c)) - Q_g(c)D^2(Q_t(c))z.$$

By (30) and (44) we have

$$D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) + Q^2(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) + k_g(Q_t(c)) \\ = D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) + 2Q_t(c)D_2(Q_g(c)) - k_t(Q_g(c)).$$

Therefore $Q_t(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) = 2Q_t(c)D^2(Q_g(c))$ and

$$(47) \quad Q_t(c)D^2(Q_g(c)) = -D(Q_t(c))D(Q_g(c)).$$

Replacing t by $(ct)_{1/2}$ we have $cQ_t(c)D^2(Q_g(c)) = -cD(Q_t(c))D(Q_g(c)) - Q_t(c)D(Q_g(c))$ and therefore

$$(48) \quad Q_t(c)D(Q_g(c)) = 0.$$

We now examine the $\mathfrak{B}$ -components of the terms of $P(g, t, a, z) = 0$ for $g, t \in \mathfrak{G}$ and $a \in \mathfrak{B}$. We have

$$(49) \quad m(1, a) + m(a, 1) = 2m(1, 1)a + 2D(Q_t(c))D(D(Q_g(c))D(a)) \\ + 2D(Q_g(c))D(D(Q_t(c))D(a)) + (k_g k_t + k_t k_g)(a)$$

where $m(a, b)$ denotes the $\mathfrak{B}$ -component of $(ga)_{1/2} \cdot (tb)_{1/2}$. Since $m(a, b)$ does depend on g and t also, we will use $m_{g,t}(a, b)$ for $m(a, b)$ when there is any chance of confusion. Replacing t by $(tb)_{1/2}$ in (49) we obtain

$$(50) \quad \begin{aligned} m(1, ab) + m(a, b) &= 2m(1, b)a + 2bD(Q_t(c))D(D(Q_g(c))D(a)) \\ &+ 2bD(Q_g(c))D(D(Q_t(c))D(a)) + 2D(Q_t(c))D(b)D(a) \\ &+ k_g(b)k_t(a) + b(k_gk_t + k_tk_g)(a). \end{aligned}$$

Define

$$(51) \quad S_{g,t}(a) = m(1, a) - m(1, 1)a - 2D(Q_g(c))D(D(Q_t(c))D(a)) - k_gk_t(a)$$

for all $a \in \mathfrak{B}$. If $g = t$ the right-hand side of (51) reduces to identity (49) with $g = t$. Therefore $S_{g,g}$ is identically zero. A simple linearization gives us

$$(52) \quad S_{g,t} = -S_{t,g}.$$

Substituting (51) into (50) and letting $a = b$ we have $S_{g,t}(a^2) + 2L_gL_t(a^2) + m(a, a) + k_gk_t(a^2) = 2S_{g,t}(a)a + m(1, 1)a^2 + 4aL_gL_t(a) + 2ak_gk_t(a)$ where $L_g = D(Q_g(c))D$ and $L_t = D(Q_t(c))D$ are derivations. Interchanging g and t in this result and subtracting gives us $2S_{g,t}(a^2) + 2L_gL_t(a^2) - 2L_tL_g(a^2) + (k_gk_t - k_tk_g)(a^2) = 4S_{g,t}(a)a + 4a(L_gL_t - L_tL_g)(a) + 2a(k_gk_t - k_tk_g)(a)$. Since both $L_gL_t - L_tL_g$ and $k_gk_t - k_tk_g$ are derivations this relation reduces to $S_{g,t}(a^2) = 2aS_{g,t}(a)$. Hence $S_{g,t}$ is a derivation of $\mathfrak{B}$ into $\mathfrak{B}$.

We can now replace (50) by

$$(53) \quad \begin{aligned} m(a, b) &= m(1, 1)ab + aS_{g,t}(b) - bS_{g,t}(a) + 2aL_gL_t(b) + 2bL_tL_g(a) \\ &- 2L_g(a)L_t(b) + ak_gk_t(b) + bk_tk_g(a) - k_g(a)k_t(b). \end{aligned}$$

By setting $g = t$, $a = 1$ and $b = Q_g(c)$ in (53) we have

$$(54) \quad m_{g,g}(1, 1)Q_g(c) = 0.$$

An examination of the $(w\mathfrak{B})_{1/2}$ -components of the terms of $P(g, g, g, z) = 0$ gives us

$$(55) \quad Q_g(c)D(m_{g,g}(1, 1)) = 0.$$

Finally we compute $P((ga)_{1/2}, (tb)_{1/2}, w, z) = 0$ to get

$$(56) \quad n_{g,t}(a, b) = -aQ_g(f_t(1)b - T_t(b) + D(k_t(b)) - bQ_t(f_g(1)a - T_g(a) + D(k_g(a)))$$

where $n_{g,t}(a, b)$ is the $\mathfrak{B}z$ -component of $(ga)_{1/2} \cdot (tb)_{1/2}$. Now $P(g, g, (wa)_{1/2}, z) = 0$. Therefore $n_{g,g}(1, 1)a + 2Q_g(f_g(1)a) + 2Q_g(T_g(a)) = 0$. From (56) with $g = t$ and $a = b = 1$ we have

$$(57) \quad Q_g(T_g(a)) = -Q_g(a)f_g(1).$$

2. In the previous section we expressed the multiplications of $\mathfrak{A}$ in terms of constants and derivations. In this section we use these multiplicative properties to construct a simple power-associative algebra of degree two from an associative algebra.

Let $\mathfrak{B}$ be an associative, commutative algebra over a field $\mathfrak{F}$ of characteristic $p > 5$. Also assume that $\mathfrak{B}$ has a single nonzero idempotent 1 that is a unity quantity.

Let $\mathfrak{B}_0, \dots, \mathfrak{B}_{n-1}$ be n homomorphic images of the vector space $\mathfrak{B}$. We let $\mathfrak{Q}$ be a sum of these n vector spaces, but not necessarily the vector space direct sum. We let $z\mathfrak{B}$ be a one-dimensional module over $\mathfrak{B}$. Clearly $z\mathfrak{B}$ is a vector space over $\mathfrak{F}$ and we form the vector space direct sum $\mathfrak{A} = \mathfrak{B} + \mathfrak{Q} + z\mathfrak{B}$. We now extend the multiplication of $\mathfrak{B}$ to $\mathfrak{A}$ in such a way that $\mathfrak{A}$ remains a commutative, power-associative algebra. First we define

$$(58) \quad (za)(zb) = (zb)(za) = ab,$$

$$(59) \quad 1x = x,$$

$$(60) \quad zy = 0$$

for every a and b in $\mathfrak{B}$, every x in $\mathfrak{A}$ and every y in $\mathfrak{Q}$. The element $e = (1/2)(1 + z)$ is an idempotent. We have already defined sufficient multiplicative properties to determine an idempotent decomposition of $\mathfrak{A}$. Clearly $\mathfrak{Q} \subseteq \mathfrak{A}_e(1/2)$ and $\mathfrak{B} + \mathfrak{B}z \subseteq \mathfrak{A}_e(1) + \mathfrak{A}_e(0)$. The second part of this statement follows by consideration of $a + bz = (c + cz) + (d - dz)$ with $2c = a + b$ and $2d = a - b$. For each of the vector spaces $\mathfrak{B}_i$ and the corresponding homomorphism of $\mathfrak{B}$ onto $\mathfrak{B}_i$ we define $(g_i b)_{1/2}$ to be the image of b . Since this notation is consistent with that of the decomposition of $\mathfrak{A}$ with respect to e we will allow the confusion of the two notations.

In order to complete our definitions of the multiplications of $\mathfrak{A}$ we choose elements b_{ij} and b_i of $\mathfrak{B}$ and derivations D_{ij} and D_i on $\mathfrak{B}$ into $\mathfrak{B}$ for $i, j = 0, 1, \dots, n-1$ with the following restrictions:

$$(61) \quad D_{ij} = -D_{ji}, \quad b_{ij} = b_{ji}, \quad b_0 = 0$$

for all values of i and j and

$$(62) \quad \begin{aligned} b_i b_j &= (b_i + b_j) b_{ij} = 0, \\ b_i D_0(b_j) &= (b_i + b_j) D_0(b_{ij}) = D_i(b_j b) + D_j(b_i b) = 0, \\ b_j D_0 D_i(b) + b_i D_0 D_j(b) &= b_j D_i(b) = 0, \\ (b_i g_j + b_j g_i)_{1/2} &= 0, \quad b_i b_{0i} D_0 = -b_i D_0 D_{0i} \end{aligned}$$

for all i and j different from 0 and all $b \in \mathfrak{B}$. We now define

$$(63) \quad (g_i a)_{1/2} b = [g(ab)]_{1/2} - D_0(ab_i) D_0(b) - 2b_i a D_0^2(b) + a D_i(b) z,$$

$$(64) \quad (g_i a)_{1/2} (bz) = -[(g_i a)_{1/2} b] z + \{g_0[a D_0(b) b_i]\}_{1/2} z,$$

$$(65) \quad \begin{aligned} (g_i a)_{1/2} (g_j b)_{1/2} &= a b b_{ij} + a D_{ij}(b) - b D_{ij}(a) + a D_j D_i(b) + b D_i D_j(a) \\ &\quad - D_j(b) D_i(a) + 2a L_i L_j(b) + 2b L_j L_i(a) - 2L_j(b) L_i(a) \\ &\quad + a b_i \{D_0[D_{0j}(b) - b_{0j} b - D_0 D_j(b)]\} z + b b_j D_0[D_{0i}(a) - b_{0i} a - D_0 D_i(a)] z \end{aligned}$$

where $L_i = D_0(b_i)D_0$, $i, j = 0, \dots, n-1$, and a and $b \in \mathfrak{B}$. Since we did not restrict $\mathfrak{Q}$ to be a direct sum of subspaces it is necessary to assume that our multiplications in $\mathfrak{A}$, as defined above, are well-defined. We place two additional assumptions on $\mathfrak{A}$. If $\mathfrak{D}$ is the set of derivations consisting of D_i and D_{ij} for all i and j we assume, in the terminology of Albert [3], that $\mathfrak{B}$ is $\mathfrak{D}$ -simple; i.e., there is no nontrivial ideal $\mathfrak{I}$ of $\mathfrak{B}$ such that $\mathfrak{I}$ is $\mathfrak{D}$ -admissible. The second assumption is that for every element g in $\mathfrak{Q}$ there is a t in $\mathfrak{Q}$ such that gt is not zero.

THEOREM 7. *Every commutative, power-associative, simple algebra of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic $p \neq 2, 3, 5$ is an algebra of the type described above.*

Proof. We choose a set of elements $g_1, \dots, g_{n-1}$ in $\mathfrak{G}$ such that every element of $\mathfrak{G}$ is expressible in the form $\sum (g_i a_i)_{1/2}$ where $a_i \in \mathfrak{B}$. We translate the notation of §1 to the notation of this section by letting $\mathfrak{Q} = \mathfrak{A}_e(1/2)$, $g_0 = w$, $D_0 = D$, $b_{00} = w^2$, $b_{0i} = f_{g_i}(1)$, $D_{0i} = T_{g_i}$, $D_i = k_{g_i}$, $b_i = Q_{g_i}(c)$, $b_{ij} = m_{g_i, g_j}(1, 1)$ and $D_{ij} = S_{g_i, g_j}$ where $i, j \neq 0$. Identities (25)–(57) give us the relations (61)–(65).

If $\mathfrak{I}$ is a nontrivial ideal of $\mathfrak{B}$ that is $\mathfrak{D}$ -admissible then if $a \in \mathfrak{I}$ we have $Q_g(a)$, $f_{ga}(b)$, $\phi_g(a)$, $\phi_{ga}(b)$, $f_{gb}(a)m_{g,t}(a, b)$ and $n_{g,t}(a, b) \in \mathfrak{I}$. This is sufficient to guarantee that $\mathfrak{I} + \mathfrak{I}z + (w\mathfrak{I})_{1/2} + (\mathfrak{G}\mathfrak{I})_{1/2}$ is a proper ideal of $\mathfrak{A}$. Since this contradicts the simplicity of $\mathfrak{A}$ we have that $\mathfrak{B}$ is $\mathfrak{D}$ -simple.

Let $(wa)_{1/2} + g$ be an element of $\mathfrak{A}_e(1/2)$ such that there is no element t in $\mathfrak{A}_e(1/2)$ such that $(wa)_{1/2}t + gt \neq 0$. Choosing t to be successively $w, (wc)_{1/2}$ and $(wc^2)_{1/2}$ and considering only the $\mathfrak{B}$ -components of the resulting terms we have $w^2a + D^2(a) + f_g(1) = w^2ac + cD^2(a) - D(a) + f_g(1)c + T_g(c) = w^2ac^2 + c^2D^2(a) + 2a - 2cD(a) + f_g(1)c^2 + 2cT_g(c) = 0$. Eliminating w^2 from these equations we have $-D(a) + T_g(c) = 2a - cD(a) + cT_g(c) = 0$. Hence $a = 0$ and $f_g(1) = T_g(c) = 0$. If we multiply g by $(wb)_{1/2}$ for $b \in \mathfrak{B}$ we have $f_g(b) = \phi_g(b) = 0$ by our assumption on g . By a previous result we had that $Q_g(c)$ was a multiple of $\phi_g(c^3)$. Hence $Q_g(c) = 0$. Now $f_g(b) = T_g(b) + k_g(D(b)) = 0$ for all $b \in \mathfrak{B}$. If we substitute bc for b we have $cT_g(b) + ck_g(D(b)) + k_g(b) = 0$. Therefore $k_g(b) = 0$. We now have that $\mathfrak{C}g = \{(ag)_{1/2} : a \in \mathfrak{B}\}$. With this choice of g and for any $b \in \mathfrak{B}$ we have $f_{ga}(b) = 0$ by (35) and $\phi_{ga}(b) = 0$ since $Q_{ga}(c) = aQ_g(c)$. Also $m_{g,t}(a, b) = aS_{g,t}(b) - bS_{g,t}(a)$. But by the assumption on g and (51) we have $S_{g,t} = 0$. Therefore $m_{g,t}(a, b) = 0$ for all a and $b \in \mathfrak{B}$. Combining this result with (56) we have $(ga)_{1/2}t = 0$ for all $a \in \mathfrak{B}$ and all $t \in \mathfrak{A}_e(1/2)$. Therefore the ideal generated by g is $\{(ag)_{1/2} : a \in \mathfrak{B}\}$. This contradicts the assumption of simplicity of $\mathfrak{A}$. Hence for each $x \in \mathfrak{A}_e(1/2)$ there is an element t in $\mathfrak{A}_e(1/2)$ such that $xt \neq 0$.

THEOREM 8. *An algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ of characteristic $p \neq 2, 3, 5$ as described in identities (58)–(65) is a commutative, power-associative, simple algebra.*

Proof. It follows readily from the definition of $\mathfrak{A}$ that $\mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$ is a subalgebra of $\mathfrak{A}$. We shall show that this subalgebra is power-associative by examining $P(x, y, s, t)$ for various values in $\mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$. If $P(x, y, s, t) = 0$ for all possible choices of the variables x, y, s and t in $\mathfrak{B}, \mathfrak{B}z$ or $(g_0\mathfrak{B})_{1/2}$ we have $\mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$ power-associative. We examine the powers of $x = a + g_0$ for $a \in \mathfrak{B}$. We have $x^2 = a^2 + b_{00} + (ag_0)_{1/2} + 2D_0(a)z$, $x^3 = a^3 + 2ab_{00} - D_0^2(a) + 5aD_0(a)z + D_0(b_{00})z + [(2a^2 + b_{00})g_0]_{1/2}$ and $x^2x^2 = x^3x$. The proof of this result depends on the properties

$$\begin{aligned}
 a(bz) &= (ab)z, \\
 (az)(bz) &= ab, \\
 (66) \quad (bz)(g_0a)_{1/2} &= -aD_0(b), \\
 b(g_0a)_{1/2} &= [(ab)g_0]_{1/2} + aD_0(b)z, \\
 (g_0a)_{1/2}(g_0b)_{1/2} &= abb_{00} + aD_0^2(b) + bD_0^2(a) - D_0(a)D_0(b).
 \end{aligned}$$

If $d \in \mathfrak{B}$ and if we replace D_0 by dD_0 , b_{00} by $b_{00}d^2 + 2dD_0^2(d) - D_0(d)^2$ and g_0 by $(g_0d)_{1/2}$ we see that relations similar to those expressed in (66) hold. Therefore we can conclude that $a + (g_0d)_{1/2}$ has a unique fourth power.

Next we investigate the fourth powers of $x = az + g_0$. We have $x^2 = a^2 + b_{00} - 2D_0(a)$, $x^3 = a^3z + b_{00}az + D_0(b_{00})z - 2D_0^2(a)z + a^2 + b_{00} - [2D_0(a)g_0]_{1/2}$ and $x^2x^2 = x^3x$. Again the only multiplicative properties used were those expressed in (66). Therefore $az + (g_0b)_{1/2}$ has a unique fourth power for all a and $b \in \mathfrak{B}$. It is easily seen that $\mathfrak{B} + \mathfrak{B}z$ is associative. Hence $a + bz$ has a unique fourth power. The assumption on the characteristic and simple linearizations of these three fourth powers we have obtained give us the result that $P(x, y, s, t) = 0$ provided that in any evaluation the four values x, y, s , and t are chosen from only two of the three subspaces $\mathfrak{B}, \mathfrak{B}z$ and $(g_0\mathfrak{B})_{1/2}$. This leaves us those choices of x, y, s and t for which $x \in \mathfrak{B}, y \in \mathfrak{B}z, s \in (g_0\mathfrak{B})_{1/2}$ and t is arbitrary. Because of the linearization process we need only consider $P(a, bz, (g_0d)_{1/2}, a)$, $P(a, bz, (g_0d)_{1/2}, bz)$ and $P(a, bz, (g_0d)_{1/2}, (g_0d)_{1/2})$. Straightforward computations, which we omit, show that each of these relations is zero. Therefore $\mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$ is power-associative.

Now let $g = \sum (g_i a_i)_{1/2}$ where $a_i \in \mathfrak{B}$. The index i , or indices i and j , of this summation and all subsequent ones will run from 1 to $n - 1$. Define

$$\begin{aligned}
 b_g &= \sum a_i b_i, \\
 D_g &= \sum a_i D_i, \\
 (67) \quad b_{0g} &= \sum a_i b_{0i} - \sum D_{0i}(a_i) + \sum D_0 D_i(a_i), \\
 D_{0g} &= \sum a_i D_{0i} - \sum D_i(a_i) D_0, \\
 b_{gg} &= \sum b_{ij} a_i a_j + 2 \sum a_i D_{ij}(a_j) + 4 \sum a_j L_j L_i(a_i) \\
 &\quad - \sum D_i(a_i) D_j(a_j).
 \end{aligned}$$

From (62) and (67) we have

$$\begin{aligned} b_g^2 &= b_g b_{gg} = b_g D_0(b_{gg}) = b_g D_g(b) = D_g(b_g) = b_g D_0 D_g(b) = 0, \\ (68) \quad b_g b_{0g} D_0(a) &= -b_g D_0 D_{0g}(a), \\ (gb_g)_{1/2} &= 0. \end{aligned}$$

From (65) we have $(ga)_{1/2}(ga)_{1/2} = b_{gg} + 2aD_g^2(a) - D_g(a)^2 + 4\sum a a_i L_i a_j L_j(a) - 2\sum a_i L_i(a) a_j L_j(a)$. Now $\sum a_i L_i(a) = \sum a_i D_0(b_i) D_0(a) = D_0(b_g) D_0(a) - \sum b_i D_0(a_i) D_0(a)$. Therefore $\sum a_i L_i(a) a_j L_j(a) = L_g(a)^2$ where $L_g = D_0(b_g) D_0$. Also

$$\begin{aligned} \sum a_i L_i a_j L_j(a) &= \sum L_g a_j L_j(a) - \sum b_i D_0(a_i) D_0 a_j L_j(a) \\ &= L_g^2(a) - \sum L_g b_j D_0(a_i) D_0(a) - \sum b_i D_0(a_i) D_0 a_j L_j(a) \\ &= L_g^2(a) - \sum D_0(b_i) D_0(b_j) a_i D_0(a_j) D_0(a) - \sum b_i D_0^2(b_j) a_j D_0(a_i) D_0(a) \\ &= L_g^2(a). \end{aligned}$$

Therefore

$$(69) \quad (ga)_{1/2}^2 = b_{gg} + 2aD_g^2(a) - D_g(a)^2 + 4aL_g^2(a) - 2L_g(a)^2.$$

We also have

$$\begin{aligned} (70) \quad b(ga)_{1/2} &= g(ab)_{1/2} - D_0(ab_g) D_0(b) - 2b_g a D_0^2(b) + a D_g(b) z, \\ (bz)(ga)_{1/2} &= g_0(a D_0(b) b_g)_{1/2} - [(ga)_{1/2} b] z \end{aligned}$$

for all a and b in $\mathfrak{B}$.

We now let $g'_0 = g_0 + g$ and $a' = a - b_g D_0(a) z$ for $a \in \mathfrak{B}$. We define a derivation $D'_0(a') = [D_0(a) + D_0(b_g)^2 D_0(a) + D_g(a)]'$ and let $t = b_{00} + 2b_{0g} - b_g D_0(b_{00}) z + b_{gg} - 2b_g D_0(b_{0g}) z$. Now $(D_0 + D_0(b_g)^2 D_0 + D_g)^2 = (D + D_g)^2 + 2L_g^2$. Therefore $a' D_0'^2(a') = a(D_0 + D_g)^2(a) + 2aL_g^2(a) - b_g[aD_0^3(a) + aD_0^2 D_g(a) + D_0(a) D_0^2(a)] z$ since $3b_g D_0 L_g^2(a) = 3b_g D_0^2 b_g D_0^2(b_g) D_0(a) = -3D_0(b_g) D_0(b_g) D_0^2(b_g) D_0(a) = 2D_0(b_g) b_g D_0^3(b_g) D_0(a) = 0$. Also $[(D_0 + D_0(b_g)^2 D_0 + D_g)(a)]^2 = [(D_0 + D_g)(a)]^2 + 2L_g(a)^2$. Therefore $[D'_0(a')]^2 = [(D_0 + D_g)(a)]^2 + 2L_g(a)^2 - 2b_g D_0^2(a) D_0(a) z$. We have, using these results, that $(g'_0 a')_{1/2}^2 = (g'_0 a)_{1/2}^2 = b_{00} a^2 + 2aD_0^2(a) - D_0(a)^2 + 2a^2 b_{0g} + 2aD_g D_0(a) + 2aD_0 D_g(a) - 2D_g(a) D_0(a) - ab_g a D_0(b_{00}) z - 2b_{00} D_0(a) ab_g z - 2ab_g D_0^3(a) z + b_{gg} a^2 + 4aL_g^2(a) - 2L_g(a)^2 + 2aD_g^2(a) - D_g(a)^2 - 2b_g a^2 D_0(b_{0g}) z - 2ab_g b_{0g} D_0(a) + 2ab_g D_0 D_{0g}(a) z - 2b_g a D_0^2 D_g(a) z = t(a^2)' + 2a'D'^2(a') - D'(a')^2 + 2b_g b_{0g} a D_0(a) + 2b_g a D_0 D_{0g}(a) = t(a^2)' + 2a'D'^2(a') - D'(a')^2$. Since $t = g_0'^2$ we have

$$(71) \quad (g'_0 a')_{1/2}^2 = g_0'^2 + 2a'D_0'^2(a) - D_0'(a')^2.$$

From (68) and (70) we have

$$\begin{aligned}
 a'(b'z) &= (a'b')z = (ab)'z, \\
 (a'z)(b'z) &= a'b' = (ab)', \\
 (b'z)(g_0a')_{1/2} &= -a'D'_0(b'), \\
 b'(g_0a')_{1/2} &= [(a'b')g'_0]_{1/2} + a'D'_0(b')z.
 \end{aligned}
 \tag{72}$$

If $\mathfrak{B}'$ is the set of all elements of the form a' where $a \in \mathfrak{B}$ then $\mathfrak{B}' + \mathfrak{B}'z + (g'_0\mathfrak{B}')_{1/2}$ is a subalgebra with multiplications similar to those expressed in (66). Hence we can conclude that this subalgebra is power-associative and that $a' + b'z + (g'_0d')_{1/2}$ has a unique fourth power for every a', b' and $d' \in \mathfrak{B}'$. But $\mathfrak{C} = \mathfrak{B}' + \mathfrak{B}'z$. Therefore $a + bz + (g_0d + gd)_{1/2}$ has a unique fourth power for every $a, b, d \in \mathfrak{B}$ and every g . If d is nonsingular then d can be absorbed in the coefficients a_i of g_i in the expression for g . Hence $a + bz + (g_0d)_{1/2} + g$ has a unique fourth power if d is nonsingular. We can restate this as $x = g_0 + \alpha(a + bz) + \beta(g_0d)_{1/2} + \gamma g$ has a unique fourth power for d a singular element of $\mathfrak{B}$, $a, b \in \mathfrak{B}$, $g = \sum (a_i g_i)_{1/2}$ and $\alpha, \beta \in \mathfrak{F}$. The characteristic is sufficiently high so that the attached polynomials of the expression $x^2x^2 - x^4$ are all zero [6]. The sum of those polynomials with a coefficient $\alpha^i\beta^j\gamma^k$ where $i + j + k = 4$ is of course also equal to zero. But by replacing α, β and γ by 1 in this sum we get $y^2y^2 - y^4 = 0$ where $y = (a + bz + (g_0d)_{1/2} + g)$. Hence any element of $\mathfrak{A}$ has a unique fourth power and $\mathfrak{A}$ is power-associative.

To complete the proof it remains only to show the simplicity of $\mathfrak{A}$. Let $\mathfrak{I}$ be a proper ideal of $\mathfrak{A}$ with the nonzero element $a + bz + t$ where $a, b \in \mathfrak{B}$ and $t \in \mathfrak{L}$. Since $z\mathfrak{I} \subseteq \mathfrak{I}$ we have $az + b \in \mathfrak{I}$. Now multiply $az + b$ by g_0 to get $(ag_0)_{1/2} + D_0(a)z - D_0(b) \in \mathfrak{I}$. By the above $(ag_0)_{1/2} \in \mathfrak{I}$. Multiplying this element by cz we get $a \in \mathfrak{I}$ and therefore $b, t, D(a)$ and $D(b) \in \mathfrak{I}$. Let $\mathfrak{P}$ be the set of all elements of $\mathfrak{B}$ that are in $\mathfrak{I}$. Clearly, $\mathfrak{P}$ is a proper ideal of $\mathfrak{B}$. Since $\mathfrak{P}\mathfrak{L} \subseteq \mathfrak{I}$ and $(\mathfrak{P}\mathfrak{L})_{1/2}\mathfrak{L} \subseteq \mathfrak{I}$ it can be easily shown that $\mathfrak{P}$ is $\mathfrak{D}$ -admissible. Hence $\mathfrak{P} = 0$ and the only nonzero elements that could be in $\mathfrak{I}$ are of the form t where $t \in \mathfrak{L}$. But by the assumption on $\mathfrak{A}$ there is an $x \in \mathfrak{L}$ such that $gx \neq 0$. Since $gx \in \mathfrak{B} + \mathfrak{B}z$ and $\mathfrak{I} \cap (\mathfrak{B} + \mathfrak{B}z) = 0$ we must have $\mathfrak{I} = 0$. Therefore $\mathfrak{A}$ is simple.

To further characterize the algebra $\mathfrak{A}$ and its subalgebra $\mathfrak{B}$ we quote a result of Harper [5, Theorem 1].

THEOREM 9. *Let $\mathfrak{B}$ be a commutative, associative algebra with unity 1 over an algebraically closed field $\mathfrak{F}$, and let $\mathfrak{B}$ be $\mathfrak{D}$ -simple relative to a set of derivations of $\mathfrak{B}$ over $\mathfrak{F}$. Then $\mathfrak{B} = \mathfrak{F}[1, x_1, \dots, x_n]$ is an algebra with generators $x_1, \dots, x_n$ over $\mathfrak{F}$ which are independent except for the relations $x_1^p = \dots = x_n^p = 0$ where p is the characteristic of $\mathfrak{F}$.*

3. Let p be a prime $\neq 2, 3, 5$ and let $\mathfrak{B}$ be the associative commutative algebra of all polynomials $\sum_{i=0}^{p-1} \alpha_i c^i$ in c with $c^p = 0$ and $c^0 = 1$, the identity of $\mathfrak{B}$. Let $\mathfrak{L}$ be $\{(g_0a)_{1/2} : a \in \mathfrak{B}\}$. Then $\mathfrak{A} = \mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$. Let $b_{00} = 0$ and D_0 be ordinary polynomial differentiation; i.e., $D_0(c) = 1$. Assume that $u = a + bz$

+ $(g_0d)_{1/2}$, where $a, b, d \in \mathfrak{B}$, is an idempotent of $\mathfrak{A}$ that is not in $\mathfrak{C}$. Then $a^2 + b^2 + 2dD_0^2(d) - D_0(d)^2 - 2dD_0(b) + 2abz + 2dD_0(a)z + 2(g_0(da))_{1/2} = a + bz + (g_0d)_{1/2}$. Therefore $d(2a-1) = 0$ and $2ab + 2dD_0(a) = b$. If $d=0$ then $u \in \mathfrak{C}$. By our assumptions $d \neq 0$ and we must have $2a-1$ is singular. Therefore we can write $a = 1/2 + c^t s$ where s is a nonsingular element of $\mathfrak{B}$ and $t \geq 1$. We have $dc^t = 0$ and $c^t b + tc^{t-1}d = 0$. Hence $c^{t+1}b = 0$. Since

$$(73) \quad a^2 + b^2 + 2dD_0^2(d) - D_0(d)^2 - 2dD_0(b) = a$$

it follows that $a^2 c^{t+1} = ac^{t+2}$. But this implies that $c^{t+1} = 2c^{t+1}$. Hence $t+1 \geq p$. Assume $t = p-1$; then $c^{p-1}b = c^{p-2}d$. Now if $b = \sum_0^{p-1} \beta_i c^i$ and $d = \sum_0^{p-1} \alpha_i c^i$ then we must have $\alpha_0 = 0$ and $\beta_0 = \alpha_1$. From (73) we must also have $\beta_0^2 - \alpha_1^2 = 1/4$ which is a contradiction. Therefore $t+1 > p$ and $a = 1/2$.

Let $x' = a' + b'z + (g_0d')_{1/2}$ be an arbitrary element of $\mathfrak{A}$. By considering the product $x'u$ we see that a necessary and sufficient condition that $x' \in \mathfrak{A}_u(1)$ is that

$$(74) \quad \begin{aligned} 2a'd &= d', \\ 2ba' + 2D_0(a')d &= b'. \end{aligned}$$

The correspondence $a' \rightarrow a' + 2a'bz + 2D_0(a')dz + 2[g_0(a'd)]_{1/2}$ is clearly a 1-1 correspondence between $\mathfrak{B}$ and $\mathfrak{A}_u(1)$ preserving the vector space operations. Therefore $\mathfrak{A}_u(1)$ is of dimension p .

If u is a stable idempotent then Albert has shown [3; 4] that $\mathfrak{A} = \mathfrak{A}_u(1) + \mathfrak{A}_u(0) + (w\mathfrak{C}') + \mathfrak{G}$ where $\mathfrak{C}' = \mathfrak{A}_u(1) + \mathfrak{A}_u(0)$ and $w\mathfrak{C}' + \mathfrak{G} = \mathfrak{A}_u(1/2)$. Albert also showed that the dimensions of $\mathfrak{A}_u(1)$, $\mathfrak{A}_u(0)$ and $w\mathfrak{C}'$ are all equal. Therefore $\mathfrak{G} = 0$. A further result of Albert's is that $\mathfrak{A}_u(1) + \mathfrak{A}_u(0) + w\mathfrak{C}'$ is associative. This implies that $\mathfrak{A}$ is a simple, associative algebra and hence we must have $c = 0$. We can conclude that our example contains no stable idempotents.

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